

On Weak Separation Property for Affine Fractal Functions.

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1. Introduction. Weak separation property (WSP), which was developed since 90-ies in papers of C.Bandt[2], K.-S.Lau and S.-M.Ngai [6] and M.Zerner[10] remains one of the main tools of analyzing dimension problems. In recent years it proved to be useful for the study of geometric structure of self-similar sets [8] and rigidity of self-similar structures [9].

In this short note we apply this notion to the theory of affine fractal interpolation funtions.

Standard definition (see [3], [4], [7]) of affine fractal function $f : [a, b] \rightarrow \mathbb{R}$ deals with a partition $a = x_0 < x_1 < \dots < x_m = b$ of the interval $[a, b]$ and a system $\mathcal{S} = \{S_1, \dots, S_m\}$ of affine transformations

$$S_i(x, y) = \begin{pmatrix} p_i & 0 \\ r_i & q_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h_i \\ s_i \end{pmatrix}, |p_i| < 1, |q_i| < 1,$$

which send vertical strip $a \leq x \leq b$ to vertical strips $L_i = \{(x, y) : x_{i-1} \leq x \leq x_i\}$. These strips divide the graph $\Gamma(f)$ to **non-overlapping** pieces $\Gamma_i = S_i(\Gamma(f)) = \Gamma(f) \cup L_i$ whose union is $\Gamma(f)$.

But a more general approach must take into account the possibility of overlaps:

For example, a system \mathcal{S} consisting of 4 maps

$$S_1 : \begin{pmatrix} 1/5 & 0 \\ 1/5 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad S_2 : \begin{pmatrix} 1/3 & 0 \\ -1/5 & -1/5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/5 \\ 1/5 \end{pmatrix},$$

$$S_3 : \begin{pmatrix} 1/3 & 0 \\ 1/5 & -1/5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 7/15 \\ 0 \end{pmatrix}, \quad S_4 : \begin{pmatrix} 1/5 & 0 \\ -1/5 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4/5 \\ 1/5 \end{pmatrix}$$

defines a self-affine function whose graph passes through the points

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$(0, 0), (1/5, 1/5), (7/15, 0), (8/15, 0), (4/5, 1/5), (1, 0)$ and has overlapping pieces

$$S_2(\Gamma) \cap S_3(\Gamma) = S_2S_4(\Gamma) = S_3S_1(\Gamma) = \Gamma \left(f|_{[\frac{7}{15}, \frac{8}{15}]} \right)$$

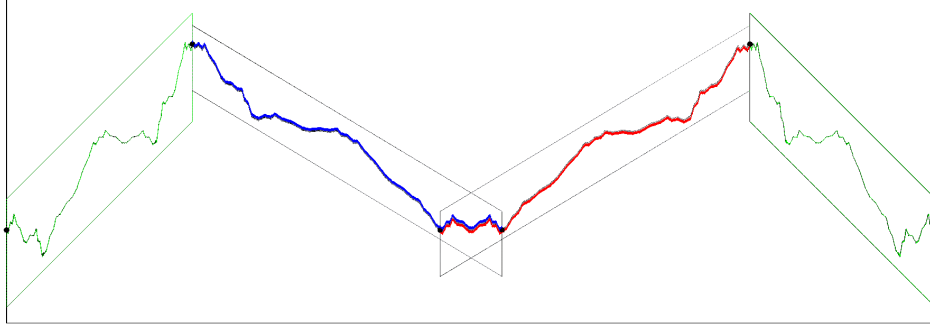


Figure 1: The graph of $f(x)$: overlapping pieces are blue and red.

In view of the above argument, we use the following definition which allows the overlaps:

Definition 1. Let $\mathcal{S} = \{S_1, \dots, S_m\}$ be a system of affine maps

$$S_i(x, y) = (p_i x + h_i, q_i y + r_i x + s_i); |p_i|, |q_i| < 1$$

A function $f(x)$ is affine fractal function on $[a, b]$ defined by the system \mathcal{S} , if its graph $\Gamma(f) = \{(x, f(x)), x \in [a, b]\}$ is the attractor of the system \mathcal{S} .

To formulate the main Theorem we recall some definitions and notation:

We denote the projections of S_i to \mathbb{R} by $S_i^\diamond(x) = p_i x + h_i$ and we denote $\mathcal{S}^\diamond = \{S_1^\diamond, \dots, S_m^\diamond\}$.

G denotes the semigroup generated by \mathcal{S} and G^\diamond denotes the semigroup generated by \mathcal{S}^\diamond .

Observe that each element $g_i = S_{i_1} S_{i_2} \dots S_{i_k}$ of the semigroup G is a map of the form $g_i(x, y) = (p_i x + h_i, q_i y + r_i x + s_i); |p_i|, |q_i| < 1$ where $p_i = p_{i_1} p_{i_2} \dots p_{i_k}$, $q_i = q_{i_1} q_{i_2} \dots q_{i_k}$, while $g_i^\diamond(x) = S_{i_1}^\diamond S_{i_2}^\diamond \dots S_{i_k}^\diamond(x) = (p_i x + h_i)$.

We define *associated families* $\mathcal{F} = G^{-1} \circ G$ and $\mathcal{F}^\diamond = G^{\diamond-1} \circ G^\diamond$ for the system \mathcal{S} (resp. \mathcal{S}^\diamond). Each element of the family \mathcal{F} is a composition $g = g_j^{-1} g_i$ and also has the form $g(x, y) = (p x + h, q y + r x + s)$, while its projection $g^\diamond = g_j^{\diamond-1} g_i^\diamond$ satisfies $g^\diamond(x) = p x + h$.

Definition 2. The system \mathcal{S} satisfies weak separation property (WSP) if Id is an isolated point in the associated family \mathcal{F} .

So, if the system \mathcal{S} does not satisfy the weak separation property, there is a sequence $g_n \in \mathcal{F}$ which converges to Id .

2. The main Theorem.

In this paper we prove the following Theorem:

Theorem 3. *Let $f(x)$ be the affine fractal function defined by a system \mathcal{S} on the segment $[a, b]$. If \mathcal{S}^\diamond does not satisfy weak separation property, then the graph $\Gamma(f)$ is a segment of a parabola.*

First of all, it follows from the definition that each fractal affine function is continuous, because its graph $\Gamma(f)$ is a compact set.

Second, a remarkable property of the maps $g \in \mathcal{F}$ is that these maps move the points of $\Gamma(f)$ along $\Gamma(f)$:

Lemma 4. *If $g \in \mathcal{F}$ and for some $x \in [a, b]$, $g^\diamond(x) \in [a, b]$, then $g(x, f(x)) = (g^\diamond(x), f(g^\diamond(x)))$.*

Proof. Let $g \in \mathcal{F}$, so $g = g_j^{-1}g_i$. If $(x, y) \in \Gamma(f)$, then $g_i(x, y) \in \Gamma(f)$. Suppose $(u, v) \in \Gamma(f)$ and $g_j^\diamond(u) = g_i^\diamond(x)$. Since $g_j(u, v) \in \Gamma(f)$, $g_j(u, v) = g_i(x, y)$, therefore $g_j^{-1}g_i(x, y) = (u, v)$, so $g(x, f(x)) = (g^\diamond(x), f(g^\diamond(x)))$. ■

These facts imply that weak separation property holds for both systems \mathcal{S}^\diamond and \mathcal{S} simultaneously:

Lemma 5. *Let $f(x)$ be the affine fractal function defined by a system \mathcal{S} on the segment $[a, b]$ whose graph is not a straight line segment. The system \mathcal{S} satisfies WSP iff \mathcal{S}^\diamond satisfies WSP.*

Proof. Suppose that WSP does not hold for \mathcal{F}^\diamond . Take three points (x_i, y_i) , $i = 1, 2, 3$ on $\Gamma(f)$ which do not lie on a line. If $g_n^\diamond \rightarrow \text{Id}$ then for each i , $g_n^\diamond(x_i) \rightarrow x_i$. Since f is continuous, $g_n(x_i, y_i) \rightarrow (x_i, y_i)$. This means that g_n converges to Id and WSP does not hold for \mathcal{F} .

Suppose now that WSP does not hold for \mathcal{F} and there is a sequence $g_n \in \mathcal{F}$ which converges to Id . Consider the coefficients of $g_n(x, y) = (p_n x + h_n, q_n y + r_n x + s_n)$: p_n and q_n converge to 1, while h_n, r_n and s_n converge to 0. Therefore $g_n^\diamond(x) = p_n x + h_n$ also converges to Id . ■

Lemma 6. *Suppose U is a family of functions $\varphi(x) \in C^3[a, b]$, satisfying inequality $|\varphi(x)| \leq M$. If for any $\varphi(x) \in U$, $\varphi''(x)$ and $\varphi'''(x)$ are monotonous and do not change the sign on $[a, b]$, then for any segment $[a', b'] \subset (a, b)$, the family $U' = \{\varphi|_{[a', b']}, \varphi \in U\}$ is bounded in $C^3([a', b'])$*

Proof. Without loss of generality, we suppose $[a, b] = [0, 1]$ and $\varphi''(x) > 0$ on $[0, 1]$. Take some $\lambda \in (2^{1/3}, 1)$.

Since $\varphi(1) \leq M$ and $\varphi(\lambda) \geq -M$, $\varphi'(\lambda) < \frac{2M}{1-\lambda}$. Similarly, we get $\varphi'(1-\lambda) > \frac{-2M}{1-\lambda}$. So $\varphi'(x) < \left| \frac{2M}{1-\lambda} \right|$ on $[1-\lambda, \lambda]$.

Repeating the same step for φ' we get $0 < \varphi''(\lambda^2) < \frac{4M}{\lambda(1-\lambda)^2}$ if φ'' increases and $\varphi''(1-\lambda^2) > \frac{4M}{\lambda(1-\lambda)^2}$ if φ'' decreases, so $\varphi''(x) < \frac{4M}{\lambda(1-\lambda)^2}$ on $[1-\lambda^2, \lambda^2]$.

The same way, we have $|\varphi'''(x)| < \frac{8M}{\lambda^3(1-\lambda)^3}$ on $[1-\lambda^3, \lambda^3]$. Taking such λ , that $[a', b'] \subset [1-\lambda^3, \lambda^3]$, we obtain the statement for the segment $[a', b']$. ■

Lemma 7. *Let $g \in \mathcal{F}$ and $\text{fix}(g^\diamond) \notin [a, b]$. Suppose that*
(i) if $x_1, x_2 \in [a, b]$ and $|x_1 - x_2| < \delta$, then $\|(x_1, f(x_1)) - (x_2, f(x_2))\| < \varepsilon$;
(ii) $\|g(x, y) - (x, y)\| < \delta$ for any point $(x, y) \in \Gamma(f)$.
Then for some $M \in \mathbb{N}$ either $\{g^n(a, f(a)), n = 0, \dots, M\}$ or $\{g^n(b, f(b)), n = 0, \dots, M\}$ is an ε -net in $\Gamma(f)$.

Proof. The condition (i) implies that if $\{x_1, \dots, x_k\}$ is a δ -net in $[a, b]$, then $\{(x_1, f(x_1)), \dots, (x_k, f(x_k))\}$ is an ε -net in $\Gamma(f)$. So we have to show that $g^{\diamond n}(a)$ or $g^{\diamond n}(b)$ form a δ -net in $[a, b]$.

Since $\text{fix}(g^\diamond) \notin [a, b]$, we have either $g^\diamond(x) > x$ for any $x \in [a, b]$ or $g^\diamond(x) < x$ for any $x \in [a, b]$.

Suppose $g^\diamond(a) > a$. Then for any point $x \in [a, b]$, $g^\diamond(x) > x$ and $g^\diamond(x) - x < \delta$. Since the limit point of the sequence $g^{\diamond n}(a)$ is outside $[a, b]$, there is such $M \in \mathbb{N}$ for which $g^{\diamond M}(a) < b < g^{\diamond M+1}(a)$, so for any $n = 1, \dots, M$, $g^{\diamond n}(a) - g^{\diamond n-1}(a) < \delta$ and $b - g^{\diamond M}(a) < \delta$. Therefore $\{g^n(a, f(a)), n = 0, 1, \dots, M\}$ is an ε -net in $\Gamma(f)$. The proof in the case $g^\diamond(b) < b$ is similar. ■

Lemma 8. *Suppose $g(x, y) \in \mathcal{F}$, $\text{fix}(g^\diamond) \notin [a, b]$ and $g^\diamond(x) > x$ on $[a, b]$. Let $g^{\diamond T}(a) = b$. Then the set $\{g^t(a, f(a)), t \in [0, T]\}$ is a graph of one of the following functions on $[a, b]$:*

1. $y = Ax^2 + Bx + C$;
2. $y = Ax + Be^{Kx} + C$;
3. $y = Ax + B(\log(x - C)) + D$, $C \notin [a, b]$.
4. $y = Ax + B(x - C)^K + D$, $C \notin [a, b]$;
5. $y = A(x - C) \log(x - C) + Dx + E$, $C \notin [a, b]$.

Proof. It is sufficient to check the statement in the case $a = 0$, $b = 1$, $f(0) = 0$ and $p > 1$. Since g is close to Id , p and q are close to 1 and therefore positive.

The five types of functions arise from direct solution of recurrence equations
 :

1. If $g(x, y) = (x + h, y + rx + s)$, then the points $g^n(0, 0)$ lie on a parabola $y = Ax^2 + Bx$, where $A = \frac{r}{2h}$ and $B = \frac{2s - hr}{2h}$;

2. If $g(x, y) = (x + h, qy + rx + s)$, $q \neq 1$, then the points $g^n(0, 0)$ lie on a graph of a function $y = Ax + B(e^{Kx} - 1)$, where $K = \frac{\log q}{h}$, $A = \frac{r}{q - 1}$, $B = \frac{hr + (q - 1)s}{(q - 1)^2}$;

3. If $g(x, y) = (px + h, y + rx + s)$, then the points $g^n(0, 0)$ lie on a graph of a function $y = Ax + B(\log(1 + x/C))$, where $C = \frac{h}{p - 1}$, $A = \frac{r}{p - 1}$, $B = \frac{hr + (1 - p)s}{(1 - p) \log p}$;

4. If $g(x, y) = (px + h, qy + rx + s)$, then the points $g^n(0, 0)$ lie on a graph of a function $y = Ax + B(x/C + 1)^K - B$, where $A = \frac{r}{p - q}$, $C = \frac{h}{p - 1}$, $B = \frac{hr + s(q - p)}{(q - 1)(q - p)}$ and $K = \frac{\log q}{\log p}$;

5. If $g(x, y) = (px + h, py + rx + s)$, then the points $g^n(0, 0)$ lie on a graph of a function $y = A(x/C + 1) \log(x/C + 1) + Bx$, where $C = \frac{h}{p - 1}$, $A = \frac{rC}{p \log p}$ and $B = \frac{Cr - s}{C - Cp}$.

Applying to x coordinate a linear transformation which sends $[a, b]$ to $[0, 1]$, we get the formulas 1-5 of the statement. ■

Proof of the Theorem 1. Take such sequence $g_n \rightarrow \text{Id}$, $g_n \in \mathcal{F}$ and such segment $[a_1, b_1] \subset (a, b)$, that for any n , $\text{fix}(g_n^\diamond) \notin [a_1, b_1]$.

Since g_n^{-1} also converge to Id , we may suppose that for any n , $p_n \geq 1$.

Without loss of generality we may suppose that for any n , $g_n^\diamond(a_1) > a_1$. Let T_n be such number, that $g_n^{T_n}(a_1) = b_1$. Each curve $\{g_n^t(a_1, f(a_1)), t \in [0, T_n]\}$ is a graph of a function $\varphi_n(x)$ on the segment $[a_1, b_1]$.

It follows from Lemma 7 that $\varphi_n(x)$ uniformly converges to $f(x)$ on $[a_1, b_1]$.

By Lemma 8, each of these functions is of one of 5 types, indicated by the Lemma. Therefore the functions $\varphi_n(x)$ have monotonous derivatives $\varphi_n''(x)$, $\varphi_n'''(x)$, which do not change their sign on $[a_1, b_1]$. By Lemma 6, for any $[a_2, b_2] \in (a_1, b_1)$, the family $\{\varphi_n(x)|_{[a_2, b_2]}\}$ is a bounded subset of $C^3([a_2, b_2])$. Therefore some subsequence of $\varphi_n(x)$ converges in $C^2([a_2, b_2])$, which implies that $f(x)$ is twice differentiable on $[a_2, b_2]$. This means that $f(x) \in C^2([a, b])$. As it was proved in [1, Theorem 3] this implies that $\Gamma(f)$ is a parabolic segment. ■

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